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Ángela Castro
Montserrat Prat
Núria Gorgorió

GOBIERNO DE ESPAÑA
MINISTERIO DE EDUCACIÓN, CULTURA Y DEPORTE

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Ángela Castro
Montserrat Prat
Núria Gorgorió
Departamento de Didáctica de la Matemática y las Ciencias Experimentales. Facultad de Ciencias de la Educación. Universidad Autónoma de Barcelona

Abstract

Research on conceptual and procedural knowledge in mathematics has been an object of interest and focus of debate throughout the years. In the literature it is possible to find discussions that address topics ranging from which should be further developed in school –the skills or the procedures– to proposals about how to study interactions between both types of knowledge. This paper analyses the current situation in the field by reviewing the most relevant characterisations in the literature for both types of knowledge, the reasons that led to changes in research focus, the current problems and the open lines of research. In turn, it contributes a summary-table of the most significant studies in the literature of each type of knowledge focusing on the mathematical domain to which they belong. The papers consulted suggest that early studies about conceptual and

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procedural knowledge focused on children, while later they spread to adolescents, young adults and pre-service teachers. Initially, research on types of knowledge mainly focused on the counting domains, single- and multi-digit addition, fractions and proportional reasoning; trying, in most cases, to determine the acquisition order of the concepts versus skills. Over the years, the interest in these two types of knowledge has increased, and its study has spread to other mathematical domains, such as equations, principles of addition and subtraction, multiplication and division. Nevertheless, it is observed in this work that, after decades of research, there is no consensus about how to define and measure the conceptual and procedural knowledge with an adequate validity level.

Keywords: conceptual knowledge, procedural knowledge, procedural flexibility, research on mathematics education, mathematical domains.

Resumen

La investigación en relación al conocimiento conceptual y al conocimiento procedimental en matemáticas ha sido tema de interés y foco de debate a lo largo de los años. En la literatura se encuentran discusiones que abordan desde qué debe desarrollarse en mayor medida en la escuela, si las habilidades o los procedimientos; hasta propuestas acerca de cómo deben estudiarse las interacciones entre ambos tipos de conocimiento. Este trabajo analiza la situación actual del campo a través de la revisión de las caracterizaciones más relevantes presentes en la literatura para ambos tipos de conocimiento, las razones que originaron cambios de enfoque en las investigaciones, las problemáticas actuales y las líneas abiertas de investigación. A su vez, se aporta un cuadro-resumen de los estudios más relevantes según cada tipo de conocimiento, poniendo el foco en el dominio matemático al que pertenecen. Las investigaciones consultadas sugieren que inicialmente los estudios sobre el conocimiento conceptual y procedimental se centraron en niños, extendiéndose posteriormente su estudio a adolescentes, adultos jóvenes y estudiantes para maestro. En un primer momento, las investigaciones sobre estos tipos de conocimiento se centraron esencialmente en los dominios de conteo, adición con uno y varios dígitos, fracciones y razonamiento proporcional; intentado, en la mayoría de los casos, determinar el orden de adquisición de los conceptos versus habilidades. Con el trascurso de los años el interés por estos dos tipos de conocimiento se ha acrecentado, y su estudio se ha extendido hacia otros dominios matemáticos, como por ejemplo, las ecuaciones, principios de adición y la substracción, multiplicación y división. No obstante, se observa en este trabajo, que tras décadas de investigación no existe un consenso acerca de cómo definir y medir el conocimiento conceptual y procedimental con un grado suficiente de validez.

Palabras clave: conocimiento conceptual, conocimiento procedimental, flexibilidad procedimental, investigación en educación matemática, dominios matemáticos.

This paper expects to foster discussion among researchers on how to approach the study of conceptual and procedural knowledge, while also recognizing the progresses and difficulties associated to it. It provides elements to guide the preparation of assessment instruments for mathematics, on a larger or a smaller scale, such as for instance the preparation of diagnostic tests for schools or university access. We should remark the need to consider the different dimensions of the knowledge to assess in the preparation of such instruments while acknowledging the existing relationships as well as the limitations in measurement.

Introduction

Different learning and cognition theories claim that our behaviour is determined by at least two types of knowledge. One provides an abstract understanding of the principles and relationships between the pieces of knowledge in a specific domain, while the other allows us to solve problems rapidly and efficiently. These types of knowledge, named respectively as conceptual knowledge and procedural knowledge by current research, have been widely acknowledged and studied throughout the years from the different mathematical domains (Baroody, 2003; Schneider & Stern, 2005). In the literature we can find several theories attempting to explain how these two types of knowledge develop and interact. These theories have been approached from different perspectives, the object of intense debate. Discussions that range from wondering what is more important to develop and encourage in school –the skills or the procedures– to how their interactions should be studied.

Early on, the study of both types of knowledge focused mainly on children (Baroody & Gannon, 1984; Canobi, Reeve & Pattison, 1998, 2003; among others). Over the years, though, this interest has increased, and adolescents and the study has incorporated young adults (Dubé, 2014; Dubé & Robinson, 2010; among others) and also pre-service teachers (Chinnappan & Forrester, 2014; Groth & Bergner, 2006; among others). Lastly, it has spread to other mathematical domains such as statistics, estimation, and equations, which were not included initially (see Table 1). However, there is no consensus in the literature to help us clearly describe what conceptual knowledge or procedural knowledge are.

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In this article we will review some of the most relevant characterizations for both types of knowledge as well as the discussions that triggered the changes in focus in research. In turn, we will present a chart with the main studies in the field according to the type of knowledge they focus on and the mathematical domain they belong to. Lastly, we will approach the current problems and the open research lines.

Characterisations of conceptual and procedural knowledge

After decades of research there seems to be no consensus to define either conceptual knowledge or procedural knowledge, or to determine the best way to measure them (Baroody, Feil & Johnson, 2007; Crooks & Alibali, 2014; Star, 2005). This is basically due to the fact that both types of knowledge are in a continuum and cannot always be separated (Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001). Perhaps visualizing these types of knowledge as a continuum – covering from scarce to richly connected knowledge – has led most researchers in mathematical education to trying to distinguish between type and quality of knowledge (Baroody, Feil & Johnson, 2007). Therefore, the diversity in the characterizations found in the field may be a reflection of the different views the researchers have on how persons acquire procedures and concepts (Star, 2005).

A revision of these characterizations suggests that conceptual knowledge is usually likened to deep, richly connected, flexible knowledge associated to significant knowledge. On the other hand, procedural knowledge has been commonly linked to scarcely connected, automatized, superficial knowledge. In this line authors such as Skemp (1978) and Bell, Costello and Küchemann (1983), among others, have provided a few approaches to the current distinction between conceptual and procedural knowledge. However the generalized use of these terms is attributed to Hiebert and Lefevre (1986), who proposed possibly one of the most renowned characterizations in the field. These authors characterized conceptual knowledge as a rich network of relationships between pieces of information that grant flexibility to access and the use of information (knowing what or why). Procedural knowledge is described by Hiebert and Lefevre (1986) as knowledge made up of two

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different parts. The first one comprises formal language or the symbolic system of representation of mathematics. The second one, comprises the algorithms or rules used to solve mathematical tasks – instructions run in a linearly predetermined sequence which determine step by step how to complete tasks (knowing how).

The view of Hiebert and Lefevre (1986) on these types of knowledge has been widely reflected on and interpreted throughout the years (Byrnes & Wasik, 1991; Carpenter, 1986; Groth & Bergner, 2006; among others). For example, we can find several characterizations sharing the characteristics described by Hiebert and Lefevre (1986), who defined conceptual knowledge in terms of the interrelations between the pieces of knowledge and the understanding of the basic concepts (Byrnes and Wasik, 1991), or the principles that rule a domain, explicitly or not (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001). It has also been linked to the integral and functional understanding of mathematical ideas; it has been highlighted that the students' degree of conceptual understanding is related to the richness and range of the connections they have established (Kilpatric, Swafford and Findelli, 2001; Rittle-Johnson & Star, 2007). These characterizations underline the flexibility of conceptual knowledge, which is unrelated to specific problems and is widely generalizable and may be verbalizable or not.

Procedural knowledge has been described as the skills to run action sequences to solve problems, as related to specific problems and non-widely generalizable (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001; Rittle-Johnson & Star, 2007). This knowledge is automatized to a certain extent, for which reason it takes minimum conscious attention to apply. Automation is attained through practice, which would grant its rapid activation and execution. In turn, this automatized character would imply that this type of knowledge is not open – or only partially open – to conscious inspection, which makes it difficult to verbalize it or transform it into higher mental processes. This would explain why it is only linked to specific problems (Schneider & Stern, 2005).

Certain authors disagree with the notion of the superficiality of procedural knowledge, traditionally characterized as operational and sequential. It has been argued that the type and quality of the knowledge must be treated as independent dimensions. Star is possibly one of the greatest advocates of the so-called reconceptualization of procedural

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knowledge (2000, 2002, 2005, 2007). Star (2005) defends the idea that the popular use of these terms confuses types of knowledge with the properties or qualities that may characterize them. This author claims that the final stage of concept acquisition occurs when the student understands the facts or principles, knows them, uses them to conduct tasks such as recognizing, identifying, explaining, assessing, or judging, among others. On the other hand, the final stage of procedure acquisition occurs when skills become routine and can be executed fluently, that is, when specific knowledge has become automatized. For this author, the term conceptual knowledge has come to cover not only the knowledge of concepts, but also the way concepts can be known. Star (2000, 2005, 2007) points to some evidence suggesting that deep procedural knowledge exists, as abstract knowledge, but not conceptual. Star (2005) proposed to define these types of knowledge by separating both dimensions, arguing that both can have a deep or superficial quality.

For Baroody, Feil and Johnson (2007), certain mathematical educators are – unintentionally – guilty of oversimplifying the use of the terms procedural knowledge or computational skills, often linking them to knowledge memorized by repetition. These authors agree with Star (2005) on the fact that these constructs should be specifically and carefully considered, while they underline that procedural depth and conceptual knowledge cannot exist by themselves. Baroody, Feil and Johnson (2007) suggest that, while partial, superficial, conceptual and procedural knowledge may exist independently, deep procedural knowledge also exists along deep conceptual knowledge and vice versa. These authors propose an outline that displays – by means of the *adaptive expertise* model – the connections or mutual dependence between both types of knowledge, as well as a reconceptualization for both constructs. It describes procedural knowledge as the mental actions or manipulations which include rules, strategies, and algorithms to complete a task; whereas conceptual knowledge is defined as the knowledge about the facts, generalizations and principles.

Possibly because of the significance of the relationships established between the pieces of knowledge characterizing conceptual knowledge, it has been linked to significant knowledge. On the other hand, because of the operationalized and sequential traits that have commonly characterized procedural knowledge, it has been linked to rote learning. However, as Rittle-Johnson and Schneider pointed out (2014), the definition of conceptual knowledge as richly connected knowledge has evolved towards to a line of thinking which has recently begun to see the richness of connections as a characteristic of conceptual knowledge that increases with experience; thus suggesting to define it in terms of the knowledge of concepts.

Describing procedural knowledge as the ability to run action sequences (procedures) to solve problems has been a predominant view in past research (Rittle-Johnson & Schneider, 2014). Although there seems to be no consensus on the existence of deep procedural knowledge, in recent years, studies have begun to approach procedural knowledge from a different perspective by incorporating the study of procedural flexibility (Star & Seifert, 2006; Rittle-Johnson & Star, 2007; Berk, Taber, Carrino & Poetzl, 2009, among others). Procedural ability has been defined as the knowledge of multiple ways to solve problems and the knowledge of when to use it (Kilpatrick et al., 2001); or as the ability to use multiple methods of solution, solving the same problem by using different methods, and choosing strategically from among the most efficient methods. Accordingly, a flexible solver knows several resolution procedures and the ability to invent or innovate in order to create new ones (Star & Seifert, 2006). Other studies have proposed distinguishing processes and procedures. In this line, Gray and Tall (1991; 1994) introduced the notion of *Procept* as the amalgam of the process and the knowledge represented by the same symbol, pointing out that the most flexible thinkers see a symbol as well as a process to make mathematics into a concept to be thought; whereas the least flexible thinkers only see it as a process to be conducted.

Conceptual and procedural knowledge in mathematics: first theories

Early research in the field approached the 'understanding versus skills' dilemma within the context of instruction programmes. Efforts were being put into determining what had to be developed and taught in school (Hiebert & Lefevre, 1986), thus affecting the organization and classification of curricular contents such as, for example, the contents and sequences of activities proposed in textbooks and instruction (e.g. see Brownell, 1935; Bruner, 1960; Gagné, 1968). The advocates of skills versus concepts claimed that the focus of instruction had to guarantee the

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memorization of computational skills, and not cultivate the mathematical understanding of concepts (e.g. Drill Theory in Baroody, 2003). On the other hand, there were others who advocated for understanding the concepts, claiming that the focus of instruction had to encourage conceptual understanding and not the memorization of basic skills (e.g. Incidental-Learning Theory in Baroody, 2003).

While some researchers devoted their efforts to deciding whether to encourage concepts or skills, others pointed towards an intermediate position where teaching mathematics should focus on the significant encouragement of skills (e.g. Meaning Theory in Baroody, 2003). Accordingly, during the last quarter of the 20th century, the interest in deciding if concepts were more important than procedures was replaced by the discussion on their order of acquisition. Two theoretical points of view ruled the debate, *concepts first* versus *procedures first* (Baroody, 2003).

Concepts-First theories suggested that children acquire or develop the understanding of concepts early on. The knowledge of concepts is used to generate and select the necessary procedures to solve problems in a specific domain (Rittle-Johnson, Siegler & Alibali, 2001; Rittle-Johnson, Siegler & Star, 2011; Schneider & Stern, 2005). The procedural knowledge of the concept is obtained from practice. These are some of the – cognitive- or curricular-oriented – empirical studies which provided evidence to support this theory: in the counting domain, the studies by Gelman and Meck (1983; 1986) and Wynn (1990); in adding and subtracting problem solving, Briars and Larkin (1984) and Riley, Greeno and Heller (1983); in single-digit addition, Siegler and Crowley (1994) and Cowan and Renton (1996), among others. On the other hand, *Procedures-First* theories suggested that the development of skills precedes and underlies the development of concepts, thus proposing that children acquire the specific skills of a specific domain early on, whereas they acquire abstract conceptual knowledge gradually from the reflection of practice (Baroody, 2003; Rittle-Johnson, Siegler & Alibali, 2001; Schneider & Stern, 2005). Studies such as Baroody and Gannon (1984); Briars and Siegler (1984); Fuson (1988); Frye, Braisby, Lowe, Maroudas & Nicolls (1989); Siegler (1991); and Siegler and Stern (1988), among others, provided empirical evidence to support this theory.

Rittle-Johnson and Siegler (1998), after a literature review on the topic, concluded that, in general, despite the evidence that, in certain domains, the development of concepts precedes the development of procedures (e.g. proportional reasoning, fraction addition, and single-digit addition), in other domains skills precede concepts (e.g. counting and multiplication of fractions). Rittle-Johnson and colleagues (Rittle-Johnson & Alibali, 1999; Rittle-Johnson and Siegler, 1998; Rittle-Johnsons, Siegler & Alibali, 2001) pointed out that *Concepts-First* theories and *Procedures-First* theories were wrong, and they criticized the claims about one type of knowledge preceding the other. These authors pointed to three factors threatening the validity of the results obtained in these studies which could explain these differences: the nature of early knowledge, the impact of previous experience, and methodological limitations. Rittle-Johnson and colleagues concluded that there is no fixed order in the development of skills as compared to the understanding of concepts. This view contributed to a change of approach in the study of conceptual and procedural knowledge.

Relationships between conceptual and procedural knowledge: the change of approach

Different authors have argued that conceptual knowledge and procedural knowledge do not develop independently from one another. Early on these two types of knowledge were seen as separate entities, either competing for attention for instruction or coexisting as separate neighbours (Hiebert & Lefevre, 1986). Over the years, the debate about them shifted to a more moderate point of view, where procedures and concepts can be related. This brought about a change of direction in the way to approach research about these types of knowledge and their development.

The iterative model

For Rittle-Johnson and Alibali (1999), conceptual knowledge and procedural knowledge do not develop independently, thus suggesting the existence of bi-directional relationships between them which develop gradually. Rittle-Johnson, Siegler and Alibali (2001) argue that these types of knowledge influence one another during their development. Accordingly they put forth the existence of an iterative model where increases in one type of knowledge lead to increases in the other, which triggers new knowledge in the former. Under this model, an increase in conceptual knowledge may lead to generating processes; while procedural knowledge may lead to profits in conceptual understanding,

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which may subsequently lead to further (not necessarily equivalent) increases in the former.

The iterative model is supported on studies in the domain of arithmetic (e.g. Baroody & Ginsburg, 1986; Byrnes, 1992; Byrnes & Wasik, 1991; Canobi, 2009; Canobi & Bethune, 2008; Hiebert & Wearne, 1996), mathematical equivalence (e.g. Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Siegler, 1998), and fractions (e.g. Rittle-Johnson, Siegler & Alibali, 2001), among others. In these studies, it is stressed that the link between these two types of knowledge can benefit the acquisition and application of both types of knowledge (Baroody, 2003). Accordingly, the significant construction of relationships between conceptual knowledge and procedures – by means of instruction sequences and/or from previous knowledge – can benefit procedural knowledge as it helps remember the procedures, it favours its effective use, it allows us to adapt the existing procedures according to the problem's demands, and it influences the generation of new procedures. For its part, procedural knowledge can influence on conceptual understanding, thus improving problem representation, increasing the availability of mental resources, the identification of wrong concepts, and the reflection of the causes that make procedures work (Baroody, 2003; Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999; and Rittle-Johnson, Siegler & Alibali, 2001).

These studies have supplied key ideas on the potential of the relationships between both types of knowledge, but they have also revealed that the way this relationship develops is unclear.

Studies focused on the relationships between both types of knowledge

Describing how conceptual and procedural knowledge interact is essential to understand how the development of knowledge occurs, as well as to improve instruction (Rittle-Johnson & Schneider, 2014).

Early on, studies on these types of knowledge focused mainly on the domains of counting, single- and multiple-digit addition, fractions, and proportional reasoning; while trying mostly to determine the order of acquisition of concepts versus skills (see Rittle-Johnson & Siegler, 1998). Over the years, the focus of the studies has spread to other mathematical domains such as equations; equivalence; principles of addition, subtraction, multiplication, and division; and to a lesser extent, domains such as estimation, statistics and density (see Table I).

TABLE I. Studies on conceptual and procedural knowledge by mathematical domain.

Domain	Conceptual K.	Procedural K.	Both
General	Crooks and Alibali (2014)	Star (2005; 2007)	Baroody (2003) Baroody, Feil and Johnson (2007) Bisanz and LeFevre (1992) Hiebert and LeFevre (1986) Carpenter (1986) Rittle-Johnson and Siegler (1998) Rittle-Johnson and Schneider (2014) Star (2000) Star and Stylianides (2000) Verschaffel, Luwel, Torbeyns and Van Dooren (2009)
Counting	Crooks and Alibali (2014)		Baroody and Ginsburg (1986) Briars and Siegler (1984) Frye, Braisby, Lowe, Maroudas, and Nicholls (1989) Gelman and Meck (1983; 1986) Lefevre et al., (20 06) Rittle-Johnson and Siegler (1998) Siegler (1991) Wynn (1990)
Single-digit and/or multiple-digit addition / subtraction		Fayol and Thevenot (2012)	Baroody and Ginsburg (1986) Canobi (2009) Canobi and Bethure (2008) Hiebert and Wearne (1996) Peled and Segalis (2005) Rittle-Johnson and Siegler (1998) Siegler and Araya (2005)
Properties of addition and/or subtraction	Baroody and Lai (2007) Canobi, Reeve and Pattison (2002) Crooks and Abali (2014) Dubé and Robinson (2010) Robinson and Dubé (2009) Robinson and Ninowski (2003) Robinson, Ninowski and Gray (2006)	Siegler and Stern (1998)	Baroody and Gannon (1984) Baroody and Ginsburg (1986) Baroody, Torbeyns and Verschaffel (2009) Baroody, Lai, Li and Baroody (2009) Bisanz, Watchorn, Piatt and Sherman (2009) Bryant, Christie and Rendu (1999) Canobi (2004; 2005; 2009) Canobi and Bethure (2008) Canobi, Reeve and Pattison (1998; 2003) Cowan and Renton (1996) Dubé (2014) Farrington-Flint, Canobi, Wood and Faulkner (2007) Gilmore and Bryant (2006; 2008) Gilmore and Spelke (2008) Gilmore and Papadatou- Pastou (2009) Patel and Canobi (2010) Prather and Alibali (2009) Robinson and LeFevre (2012) Siegler and Araya (2005) Siegler and Crowley (1994) Schneider (2012) Schneider and Stern (2009)

Domain	Conceptual K.	Procedural K.	Both
Equations		Rittle-Johnson, Star and Durkin (2012) Star (2002) Star and Newton (2009) Star and Seifert (2006)	Prather and Alibali (2011) Rittle-Johnson and Star (2007; 2009) Rittle-Johnson, Star and Durkin (2009) Schneider, Rittle -Johnson and Stern (2011)
Fractions	Hech, Close and Santini (2003)		Bailey et al.,(2015) Byrnes and Wasik (1991) Chinnappan and Forrester (2014) Hallett, Nunes and Bryant (2010) Hallett, Nunes, Bryant and Thorpe (2012) Peled and Segalis (2005) Rayner, Pitsolantis and Osana (2009) Rittle-Johnson and Siegler (1998) Rittle-Johnson, Siegler and Alibali (2001) Schneider and Stern (2005; 2010)
Equivalence	Crooks and Alibali (2014)		Matthews and Rittle-Johnson (2009) Rittle-Johnson and Alibali (1999)
Problem Solving	Oyarzún and Salvo (2010)	Briars and Larkin (1984)	Blöte, Van der Burg and Klein (2001) Riley, Greeno and Heller (1983)
Density	Schneider and Hardy (2000)		
Multiplication and division	Dubé and Robinson (2010a, b) Robinson and Ninowski (2003) Robinson, Ninowski and Gray (2006)	Fayol and Thevenot (2012) Rittle-Johnson and Kmicikewyck (2008)	Dubé (2014) Robinson and LeFevre (2012)
Decimals			Peled and Segalis (2005) Rittle-Johnson and Koedinger (2009)
Statistics			Groth and Bergner (2006)
Estimation			LeFevre, Greenham and Waheed (1993) Star and Rittle-Johnson (2009)
Proportional Reasoning		Berk, Taber, Carrino and Poetzl (2009)	Dixon and Moore (1996) Rittle-Johnson and Siegler (1998)
Algebra		Blessing and Anderson (1996)	
Mental Calculation			Blöte, Klein and Beishuizen (2000)
Integers			Byrnes (1992)

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Concept-procedure interactions have been analysed from two perspectives mainly: problem solving and instruction sequences. There are also research – to a lesser extent – studying the development of these types of knowledge individually, either acknowledging and considering the existence of the bonds between them (e.g. Blöte, Klein & Beishuinzen, 2000; Hech, Close & Santini, 2003; Star & Seifter, 2006) or not (e.g. Blessing & Anderson, 1996).

From the perspective of problem solving in the domain of the principles of addition, relationships have been analysed between the procedure skills in addition problem solving and the conceptual knowledge of some principles of addition such as commutativity, associativity, and inversion, among others (e.g. Canobi, 2004, 2005; Canobi & Bethune, 2008; Canobi, Reeve & Pattison, 1998, 2003; Farrington-Flint, Canobi, Wood and Faulkner, 2007; Gilmore & Bryant, 2006; 2008; Pastel & Canobi, 2010). In these studies, conceptual knowledge has been evaluated through task justification and judgment and through the use of part-whole relationships to solve problems. The problem-solving skill has been evaluated through the repertoires of procedures, speed and precision. Different conceptual comprehension and arithmetic skill profiles in children have originated from this trying to account for the way this process occurs. In light of the results obtained, the diversity of knowledge profiles identified in children has been interpreted as a sign of the complexity in the part-whole relationship children understand, and the existence of different routes of development (Canobi, 2004; 2005; Canobi, Reeve & Pattison, 1998, 2003). It has been pointed out that children sometimes develop conceptual comprehension without the competent skill, and that these differences suggest that, for most children, conceptual comprehension and computational skills develop together, while for others it does not (Gilmore & Bryant, 2006, 2008; Gilmore & Papadatpu-Pastou, 2009). It has also been suggested that the conceptual comprehension of children and their procedural skills are – to a large extent – unrelated, at least in certain points during its development (Schneider & Star, 2009).

Studies evaluating the implementation of instruction sequences focus their attention on the way a learning cycle – usually focused on one type – might facilitate the change of the other type of knowledge. In this line, the following aspects have been analysed: the effects of instruction on the understanding of multiple-digit numbers and computational skills

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(Hiebert & Wearne, 1996); the effects of instruction based conceptually on the development of the flexibility of addition and subtraction procedures (Blöte, Van der Burg & Klein, 2001); and the benefits of an iterative sequence – as compared to a concepts-before-procedures sequence – on decimal positional value and arithmetic procedures (Rittle-Johnson & Koedinger, 2009); among others. Results in these types of studies generally suggest that sequences are a good source to foster the relationships between both types of knowledge and thus facilitate their study. It is suggested that sequenced problem practice conceptually based on principles of addition improves the children's capacity to enlarge their processual learning to the new problems and it leads to a better capacity to verbalize key concepts (Canobi, 2009). It is also suggested that an iterative sequence between concepts and procedures can encourage an increase of correct procedures, lead to a better retrieval of knowledge and support their integration (Rittle-Johnson & Koedinger, 2009). While knowledge in these types of studies tends to be evaluated with pre- and post-test tasks, it may occasionally be necessary to use some type of previous knowledge to fully benefit from the other's lessons (Rittle-Johnson & Koedinger, 2009). It is important to consider that, when assessing the interactions between conceptual and procedural knowledge through instruction sequences, it is difficult to do so independently.

Current research and open lines

Despite breakthroughs in the field regarding the study of conceptual and procedural knowledge, the review we have conducted points to the existence of some disagreements which are still the focus of debate, such as the difficulty to define and measure both types of knowledge. The variety of characterizations found in the literature and the tasks used to measure them, which do not always agree with how these types of knowledge have been defined, makes it difficult to understand the main findings, the ways these types of knowledge interrelate, and to determine the most efficient way to use research to guide teaching practices (Crooks & Alibali, 2014).

Even though conceptual and procedural knowledge cannot always be separated, it is useful to distinguish them in order to understand the development of knowledge better (Rittle-Johnson & Schneider, 2014).

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Conceptual knowledge characterized as rich, complex knowledge has been evaluated through a large number of tasks including the use of both explicit – such as the concepts' definitions – and implicit conceptual knowledge indicators – such as evaluation, judgement, justification and application of procedures (Canobi, 2005; Canobi & Bethure, 2008; among others). The evaluation of procedural knowledge – traditionally characterized as operationalized, sequenced knowledge – has become relatively standardized, whereas there are fewer tasks to measure it (Rittle-Johnson & Schneider, 2014). To evaluate procedural knowledge, the participants usually solve a set of problems evaluated according to speed, precision or the repertoire of specific procedures to obtain their responses (Canobi, Reeve & Pattison, 1998; LeFevre et al., 2006; Schneider & Stern, 2010, among others). The participants are familiar with most procedural tasks, whereas only certain times do they comprise the acknowledgment of an unknown procedure and pertinent adaptations, or variations to face the new problem (Rittle-Johnson & Schneider, 2014).

The view of researchers regarding conceptual and procedural knowledge may have important repercussions on the results obtained, as they can affect the nature of the relationships between both types of knowledge (Gilmore & Papadatpu-Patsou, 2009). For Rittle-Johnson and Schneider (2014), no standardized approaches have so far been developed to evaluate these types of knowledge with proven validity, reliability and objectivity. We agree with these authors on the fact that this is a difficult situation as the knowledge stored in memory must be deduced from conduct. However, human behaviour comes from the complex interaction of numerous cognitive processes and it does not generally reflect the content of memory reliably. Future studies should carefully consider these constructs, and they should examine the way these types of knowledge are operationalized and measured while providing evidence of the validity of the measures and specifying more thorough models for better understanding (Baroody, Feil & Johnson, 2007; Rittle-Johnson & Schneider, 2014; Star, 2007).

Crooks and Alibali (2014) made an effort to offer a characterization of conceptual knowledge comprising a large part of the previous characterizations. These authors divided conceptual knowledge into two types of knowledge and established evaluation indicators for both of them. They consider conceptual knowledge as: (i) the knowledge of general principles, which comprises rules, definitions, connections and

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the aspects related to the domain's structure; and (ii) the knowledge of the principles underlying procedures, which involves knowing why certain procedures work for certain problems, what the purpose of each step of a procedure is, knowing the links between these steps and their conceptual fundamentals.

Another question to bear in mind is the lack of studies in other mathematical domains as well as in other educational levels (Star, 2000). Research is still mostly conducted on primary school. In recent years, adolescents, young adults and pre-service teachers have begun to be included in research. However, further research is needed in the field so that it is possible to contrast the results obtained and achieve useful theory to help untangle the complexity of these types of knowledge and their relationships.

Lastly, we would like to highlight the progress achieved in understanding the development of these types of knowledge in the last 15 years. For Rittle-Johnson and Schneider (2014), it would be an important step to develop a wider model of the relationships between concepts and procedures while delving into the way these types of knowledge are stored independently in long-term memory; how this change occurs with experience; how age and individual differences impact the relationships between these types of knowledge; and the study of the efficacy of the different teaching methods; among other aspects.

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Contact address: Departamento de Didáctica de la Matemática y las Ciencias Experimentales. Facultad de Ciencias de la Educación. Universidad Autónoma de Barcelona, 08193 Bellaterra (Barcelona), España. E-mail: a.castro.inostroza@gmail.com

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