

revista de
EDUCACIÓN

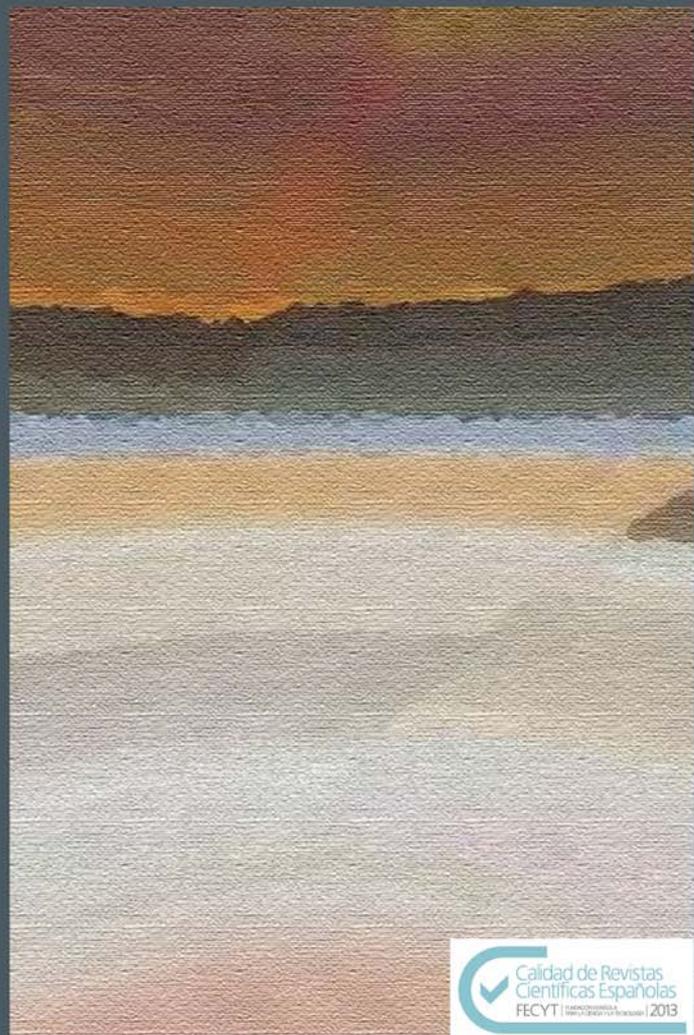
Nº 370 OCTUBRE-DICIEMBRE 2015



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sobre razonamiento algebraico elemental de futuros
maestros**

**Assessing the didactic – mathematical knowledge of
prospective primary school teachers on elementary
algebraic reasoning**

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Assessing prospective primary school teachers' didactic-mathematical knowledge on elementary algebraic reasoning¹

Evaluación de conocimientos didáctico-matemáticos sobre razonamiento algebraico elemental de futuros maestros

DOI: 10.4438/1988-592X-RE-2015-370-303

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Abstract

Different studies suggest the need to introduce elementary algebraic reasoning at an early age, which in current curricular proposals is only taught in Secondary Education. In this paper, the responses to a questionnaire evaluating prospective primary school teachers' didactic-mathematical knowledge regarding elementary algebraic reasoning are analysed. The aim is to provide a diagnosis of prospective primary school teachers' knowledge regarding elementary algebra and its teaching. The results will be used to develop a training programme for teachers,

⁽¹⁾ The research reported in this article was carried out as part of the following projects: EDU2012-31869 and EDU2013-41141-P, Ministry of Economy and Competitiveness (MINECO, Spain). It has been also sponsored by FCT/MEC through national funds (PIDDAC) and co-sponsored by FEDER through COMPETE - Programa Operacional Fatores de Competitividade within the PEst-C/CED/UI0194/2013 project.

which ultimately ensures an effective study process in primary education. The sample consisted of 597 students from the Universities of Granada, Jaén, Public University of Navarra, Santiago de Compostela in Spain, and the University of Aveiro in Portugal. The questionnaire comprises 25 items that assess algebraic and pedagogical content knowledge in primary education. Taking into account that in these universities the groups are *ad extra* homogeneous and *ad intro* heterogeneous, entire groups were intentionally selected at each university. The quantitative analysis of the results enabled the estimation of the instrument psychometric properties (difficulty and discrimination indexes, reliability and validity). The comparison of the training programmes in mathematics and the way in which the subject is taught in the different universities reveals that current curriculum emphasizes on the psycho-pedagogical area. It also reveals an unsatisfactory disciplinary training that does not include algebraic reasoning. Results show significant differences among universities in the participants' level of knowledge in the various didactic - mathematical knowledge components. We conclude that it is necessary to improve the teachers' education programmes so they can progressively foster their pupils' algebraic reasoning. To achieve this goal these programmes should include specific training activities on elementary algebraic reasoning.

Keywords: teacher training, evaluation, algebraic knowledge, didactical knowledge, comparative study

Resumen

Diferentes estudios sugieren la necesidad de incorporar desde edades tempranas el razonamiento algebraico elemental, que en las propuestas curriculares clásicas se inicia en la Educación Secundaria Obligatoria. En este trabajo se analizan los resultados de aplicar un cuestionario de evaluación de conocimientos didáctico - matemáticos sobre razonamiento algebraico elemental a una muestra de estudiantes del Grado en Maestro en Educación Primaria. El objetivo es la elaboración de un diagnóstico sobre la competencia algebraica elemental y su didáctica de los futuros maestros, que permita enmarcar un programa formativo para estos, que garantice finalmente procesos de estudio efectivos en la educación primaria. La muestra está compuesta por 597 estudiantes de las Universidades de Granada, Jaén, Pública de Navarra, Santiago de Compostela en España y de Aveiro en Portugal. En estas universidades, los grupos se constituyen homogéneos *ad extra* y heterogéneos *ad intro*. Se propone pues la selección intencional de grupos completos en cada universidad. El cuestionario consta de 25 ítems que evalúan tanto conocimientos algebraicos como conocimientos sobre la enseñanza y aprendizaje del álgebra en Educación Primaria. El análisis cuantitativo de los resultados ha permitido explorar las características psicométricas del instrumento (índices de dificultad, discriminación, fiabilidad y validez). La comparación de los programas de formación en matemáticas y su didáctica entre las distintas universidades participantes revela el énfasis psicopedagógico del Plan de estudios vigente y

muestra una formación disciplinar deficiente, que, en particular, no incluye el bloque de razonamiento algebraico. Los resultados muestran un bajo nivel de conocimientos generalizado en las distintas componentes del conocimiento didáctico - matemático, con diferencias significativas entre las universidades. Se concluye que es necesario revisar los programas de formación y planificar el diseño de acciones formativas específicas sobre los contenidos algebraicos elementales, a fin de capacitar a los futuros maestros para que puedan promover en los alumnos de primaria el progresivo desarrollo del pensamiento algebraico.

Palabras clave: formación de maestros, evaluación, conocimiento algebraico, conocimiento didáctico, estudio comparado.

Introduction

Teacher training in mathematics and didactics is essential for the improvement of both teaching and learning. Therefore, it is a field of research and development that has been included in mathematics education. In Spain, this training occurs in the Faculty of Education and it is affected by a changing regulatory environment, as well as by the lack of human, technological and time resources. However, the implementation of a Teacher's Degree in Primary Education (TDPE) in compliance with the framework of the European Higher Education Area (EHEA) and the development of research in the departments of specific didactics are opening new possibilities for improvement and development of teacher training programmes.

This paper presents the results of a research project on "Evaluation and development of prospective primary school teachers' didactic-mathematical knowledge on elementary algebraic reasoning". The objective is to contribute with new knowledge, teaching resources and methodological proposals for the improvement of initial training in mathematics, and its teaching at the level of Primary Education, taking into account the Spanish educational context as well as the new curricula.

This paper is closely related to the TEDS-M² study (Teacher Education and Development Study in Mathematics) (Tatto et al., 2008; INEE, 2012;

⁽²⁾ TEDS-M was the first large-scale international comparative study in higher education, focusing on initial training of primary and early secondary education mathematics teachers. The TEDS-M data were collected in 2008, so the situation described relates to the curriculum and conditions at that time. Since then Spanish universities have undergone significant changes in the curriculum for teacher training.

Sanz & Martín, 2014). Although, it is limited to a specific aspect of the didactic-mathematical training of prospective teachers, «elementary algebraic reasoning», using a sample of five institutions specialised in teacher training: four of them Spanish universities (Granada, Jaén, Public University of Navarra, Santiago) and a Portuguese university (Aveiro).

Algebraic reasoning, perceived as mathematical modelling tool, is applicable to different content blocks (arithmetic, geometry, measurement, data analysis and probability). Additionally, generalisation and symbolisation processes, as well as the concepts of relation, variable, equation and function are indispensable tools in mathematics, both at a professional and at a school level. The Principles and Standards for School Mathematics of the National Council of Mathematics Teachers (NCTM, 2000) recommend the introduction of algebra in Preschool as one of five content blocks: Numbers and Operations, Geometry, Measurement, Data Analysis, and Probability.

The new Spanish curricular guidelines (MECD, 2014) encourage the development of algebraic thinking in Primary Education; namely with the intent to “achieve that all students, after completing primary education, are able to describe and analyse situations of change, find patterns and mathematical laws in numerical, geometric and functional contexts, evaluating their usefulness in prediction-making»(p. 19387). Thus, introduction to algebra is now part of the continuing nuclei, i.e., «the central aspects of mathematics teaching and learning which presumably remain unchanged over time» (Wilhelmi, 2014, 33).

Keeping this in mind, if algebra is to be included in mathematics at school as a new transversal content, teachers should embrace this new broadened vision and acquire basic skills in elementary algebraic reasoning. In previous papers, (Godino, Castro, Aké & Wilhelmi, 2012; Godino, Aké, Gonzato & Wilhelmi, 2014) we have presented preliminary results regarding this part of the project which focuses on the study of didactical problems in algebra and its didactics at the level of training in the TDPE.

In Godino et al. (2015) we present the design of a questionnaire used to evaluate relevant aspects of prospective teachers' didactic-mathematical knowledge (DMK) regarding Elementary Algebraic Reasoning (EAR). The use of the DMK-EAR questionnaire with a sample of prospective teachers provides an insight into the status of their didactic-mathematical knowledge in elementary algebraic reasoning, as well as criteria for the development of tailored training programmes.

This paper describes the mathematical and didactical knowledge of TDPE students about EAR and its relation with the training programmes they are attending; i.e. there is an approach to the issue of diagnosing prospective teachers' didactic-mathematical knowledge on EAR based on the existing training programmes.

Therefore, following this general introduction, there is a description of the theoretical framework supporting the development of the questionnaire, as well as the variables taken into account. Subsequently, there is a characterisation of the context and student sample answering the questionnaire. Then, there is a presentation of the global and partial results of the quantitative study, indicating some psychometric properties of the instrument and distinguishing algebraic and didactical knowledge. Finally, the conclusions and a discussion of the implications of the study are presented.

Theoretical Framework

This research is based on two theoretical framework components: 1) Nature of algebraic reasoning in primary education (Carraher & Schliemann, 2007; Cai & Knuth, 2011); and 2) didactic-mathematical knowledge of the Maths Teacher (Godino, 2009). These components support the evaluation instrument used in this research.

Godino, Ake, Gonzato & Wilhelmi (2014) present a model of EAR structured into four algebraisation levels. This model takes into account the objects and processes involved in mathematical activity. At level 0 there are no algebraic traits in mathematical activity, whereas level 3 is clearly algebraic. The intermediate levels 1 and 2, considered as progressive algebraisation levels, showcase some objects and processes which are algebraic in nature.

Regarding the modelling tool adopted for the teacher's didactic-mathematical knowledge³, two types of variable were taken into account: algebraic content and didactical content. In terms of the *algebraic content* three values or categories with different subcategories were taken into consideration:

⁽²⁾ TEDS-M was the first large-scale international comparative study in higher education, focusing on initial training of primary and early secondary education mathematics teachers. The TEDS-M data were collected in 2008, so the situation described relates to the curriculum and conditions at that time. Since then Spanish universities have undergone significant changes in the curriculum for teacher training.

- *Structures* (equivalence relation; properties of operations, equations,...)
- *Functions* (arithmetic patterns, geometric patterns; linear, affine, quadratic functions,...)
- *Modelling tool* (context problems solved via equations or function relationships)

Regarding the variable *didactical content* (based on an algebraic content, associated with primary school level or higher) the following categories were considered:

- *Epistemic*: recognition of objects and algebraic processes (representations, concepts, procedures, properties, generalisation, modelling); recognition of algebraisation levels.
- *Cognitive*: personal meanings inferred by the students (knowledge, understanding and competence in elementary algebraic contents); learning conflicts at the level of algebraic objects and processes.
- *Instructional*: Resources for teaching algebra in primary school (situation-problem, technical resources), as well as its adaptation to the school curriculum.

Method

Instrument

The instrument used for data collection was the DMK-EAR questionnaire (see Appendix) composed of a set of 10 tasks, each of which consists of items that evaluate different aspects of algebraic and didactic–algebraic content. The description of the questionnaire development process and the discussion of its validity and reliability are available in Godino et al. (2015).

Table I shows a classification of the different items of the questionnaire according to the categories of didactic-mathematical knowledge on EAR. This instrument will not evaluate all categories of didactic-mathematical knowledge proposed by the DMK-EAR model, as it is designed to obtain the information provided by the students in writing within a limited period of time (2h approximately).

One should take into account that the DMK-EAR categories are not disjoint or exclusive; thus, the same item may appear in more than one category. For example, item 10a), «Name a variable of the problem that enables the study of functions», involves the content «functions» but also «modelling». Additionally, the student is asked to state one problem variant with a specific didactic purpose, leading prospective teachers to apply knowledge about an «instructional resource.» Moreover, the item also involves specialised knowledge of the mathematical content itself, i.e. the epistemic facet in its situational and regulatory components (concept of function and its representations). For this reason, in table I, some items are included in more than one cell.

TABLE I. Contents evaluated by each item of the questionnaire

DIDACTICAL CONTENT	ALGEBRAIC CONTENT					
	Structures (S)		Functions (F)		Modelling (M)	
	Primary	Advanced	Primary	Advanced	Primary	Advanced
Epistemic (Algebraisation levels)	<u>EPI-E1</u> 2b; 4b	<u>EPI-E2</u> 7b;7c	<u>EPI-F1</u> 10b	<u>EPI-F2</u> 5c; 6b	<u>EPI-M1</u>	<u>EPI-M2</u> 9b
Cognitive (Personal meanings)	<u>COG-E1</u> 1a; 1b 2a; 3b; 4c	<u>COG-E2</u>	<u>COG-F1</u> 6c	<u>COG-F2</u>	<u>COG-M1</u>	<u>COG-M2</u>
Instructional (Situations and resources)	<u>INS-E1</u> 8b	<u>INS-E2</u> 8b	<u>INS-F1</u> 8b	<u>INS-F2</u> 5b; 8b; 10a	<u>INS-M1</u> 9c); 10a	<u>INS-M2</u> 8b
Algebraic Content (only common or advanced knowledge)	<u>ALG-E1</u> 4a; 3a	<u>ALG -E2</u> 7a	<u>ALG -F1</u> 5a; 10b;	<u>ALG -F2</u> 6a; 8a; 9a	<u>ALG -M1</u>	<u>ALG -M2</u> 9a

Source: Godino et al. (2015)

Dependent variables

The first objective is to define the quantitative variable «degree of accuracy of the answers given to the 25 items of the questionnaire». Partially correct answers were assessed positively, so the score assigned to each item was:

- 0 points, if the answer is incorrect.
- 1 point, if it is partially correct,
- 2 points, if it is correct.

In addition to the variable «total score», i.e. sum of scores obtained in all 25 items (0-50 points), quantitative variables are defined regarding «common and advanced mathematical knowledge» and «didactical content», representing the sum by rows or columns of the variables provided in Table I.

a) Concerning common and advanced mathematical knowledge

A1_ALG: Assess knowledge of «algebra» characteristic in Primary school (common knowledge) or Secondary school (advanced knowledge). This scale includes items 3a, 4a, 5a, 6a, 7a, 8a, 9a, and 10b.

A2_EST: Evaluates knowledge related to properties of the algebraic structures used in equation solving. It includes items 1a, 1b, 2a, 2b, 3a, 3b, 4a, 4b, 4c, 7a, 7b, 7c, and 8b.

A3_FUN: Includes knowledge related to geometric patterns and functions. It includes items 5a, 5b, 5c, 6a, 6c, 8a, 8b, 9a, 10a, and 10b.

A4_MOD: Evaluates knowledge related to algebraic modelling (using equations or functions). It includes item, 8b, 9a, 9b, 9c, and 10a.

b) Concerning didactical content

CD1_EPI: Incorporate knowledge on the epistemic facet of the DMK – EAR and includes items 2b, 4b, 5c, 6b, 7b, 7c, 9b, and 10b.

CD2_COG: Contain knowledge on the cognitive facet and includes items 1a, 1b, 2a, 3b, 4c, and 6c.

CD3_INS: Assess knowledge on the instructional facet and includes items 5b, 8b, 9c, and 10a.

The different dependent variables comprise a number of different items scoring 0, 1 or 2, so its variation range is distinct. To ease the comparison and interpretation of the scores they were converted to the interval [0, 10].

To ensure reliability of the student response coding process, a criteria protocol was developed with illustrative examples of the type of response. Additionally, ambiguous interpretation was discussed within the research group.

Explanatory variables

To interpret the results of the evaluation of student knowledge, the following explanatory variables were taken into account:

Admission rate to the Teacher's Degree in Elementary Education (ratio between the number of places available and the number of applications for admission)

Cut-off score in the admission exam.

Period of time at which the test was done: 1. it was applied at the beginning of the training process; 2. it was applied at the end of the training process.

Context and sample description

The target population of this research are Spanish students of the Teacher's Degree in Primary Education (TDPE) (MECD, 2007). The questionnaire has been used with several groups of students from the Universities of Granada, Jaén, Public University of Navarra, Santiago de Compostela, attending the TDPE under the new curriculum. The questionnaire has also been used with a sample of students from the University of Aveiro, which allows for a comparative study. Both Spanish and Portuguese curricula have been reviewed within the framework of the European Higher Education Area (EHEA).

The number of credits assigned to Mathematics Education in the Spanish curriculum guidelines is 18 (Plan 2010) for the Universities of Jaén, Public of Navarra and Santiago; in Granada the number is 22. Although the teaching course has gone from 3 to 4 years and that, therefore, the total number of credits has increased from 180 to 240, these studies continue to have a strong psychological and pedagogical component. The percentage of credits allocated to training in mathematics and its didactics is 7.5%, which is similar to the 1991-2010 Plan (6.7% according to the study Rico, Gomez & Cañadas, 2014).

In this study, it was not possible to include a detailed analysis of the contents of mathematics and its didactics offered in the five universities, as was done by Rico et al. (2014). However, we found that algebra does not appear explicitly in the syllabus of the corresponding subjects. Emphasis is on the study of numbers, measurement, geometry and data

analysis. On the other hand, at the University of Aveiro the course, which is divided into 3 years, includes several subjects that seek to intensify the mathematical training of prospective teachers. However, algebraic knowledge is, only briefly discussed in the subject Didactics of Mathematics and Educational Technology.

In the Universities of Granada and Aveiro, the questionnaire was given to last-year students. In the remaining universities the questionnaire was given to students in the beginning of the degree. Nevertheless, given that throughout the remainder of these courses there is no explicit training in EAR, results show the students' professional knowledge about EAR, i.e. that which they will be using throughout their career.

The questionnaire was used with entire classes not only to gather information enabling the evaluation of student knowledge, but also to use the task solving activity as a device for further training. The present study refers only to the evaluation results.

Table II shows the sample size in each university and the values of three possible variables explaining the different scores obtained in each university.

TABLE II. Sample distribution according to university and explanatory variables

University	Number of students	Percentage	Cut-off score		Admission rate	Moment it was used
			Average 2011-2014	Standard deviation		
Aveiro	66	11.6	6.542	0.227	0.191	2
Granada	91	15.4	5.920	0.146	0.333	2
Jaén	230	38.53	5.000	0.000	0.531	1
Pública de Navarra	129	21.61	7.070	0.233	0.316	1
Santiago	81	13.57	6.740	0.289	0.158	1
TOTAL	597	100.00				

Source: Authors. The data relating to the cut-off and access quota were taken from the following sites:

Aveiro (Direcção Geral de Ensino Superior, DGES): <http://www.acessoensinosuperior.pt/detcursopi.asp?codc=9853&code=0300>

Granada – Jaén (Junta de Andalucía): <http://www.juntadeandalucia.es/economiainnovacionciencia/sguit>

U. Public of Navarra (Universidad Pública de Navarra): <http://www.unavarra.es/estudios/acceso-y-matricula/grados/notas-de-corte>

Santiago (Comisión Interuniversitaria de Galicia): <http://ciug.cesga.es/notascorte.html>

Data analysis

For the analysis of the items, descriptive statistics techniques (mean, dispersion, and box plots) were used. After checking the accurateness of the total scores and subscales, parametric inferential tests were computed to compare groups, including Analysis of Variance. Finally, a factor analysis was performed to explore the structure of the participants' responses to the items.

Results of the quantitative study

In this section some psychometric characteristics of the DMK-EAR questionnaire are discussed. This study, which adds to the study of Godino et al. (2015), is crucial because the sample size is significantly larger (597 students) and it allows for a better adjustment of some indicators. Next, there is an analysis of the results concerning the level of knowledge on algebra and didactics of algebra, relating such knowledge to the existing learning opportunities offered by the several universities.

Some psychometric properties of the instrument

Item Analysis

Table III presents the difficulty indices of the items included in the questionnaire calculated on the entire sample. This index does not correspond to the percentage of correct responses as scores of 0, 1 and 2 were ascribed based on the degree of correction. The average of scores for each item was calculated and, to make the analysis simpler, they were converted to interval [0-100]. As indicated in Table III, the mean score was 32.5 (standard error 1.43) (on a scale of 0 to 100), indicating that the level of knowledge is insufficient. In fact, only 5 items have a difficulty index above 50, whereas the difficulty index of 21 items is below this level; in addition, 11 items have a difficulty index below 30 (at least 7 out of 10 answers is incorrect).

TABLE III. Difficulty index of the items in the DMK–EAR questionnaire (n=597)

ITEM. Descriptor	Difficulty index		Discrimination: Mean difference (P ₃₃ - P ₆₆)	
	M	Standard error	t	p
1a. Equality arithmetic result. Explanation	80.7	1.40	6.08	.000
1b. Equality arithmetic result. Interpretation	55.4	1.90	11.64	.000
2a. Equality equivalence. Explanation	69.0	1.64	8.39	.000
2b. Equality equivalence. Properties	15.5	1.20	6.08	.000
3a. Sum three numbers. Generalisation	48.0	1.61	10.08	.000
3b. Sum three numbers. Type of reasoning	30.7	1.73	8.44	.000
4a. Partial sum. Solve and explain	42.3	1.95	10.64	.000
4b. Partial sum. Algebraic solution	3.8	0.65	4.16	.000
4c. Partial sum. School solution	10.2	1.14	5.27	.000
5a. Hexagonal pattern. Two terms	57.6	1.66	8.84	.000
5b. Hexagonal pattern. Algebraic generalisation	15.0	1.25	7.98	.000
5c. Hexagonal pattern. Type of algebraic objects	6.1	0.87	5.39	.000
6a. Square pattern. General solution	42.4	1.76	14.35	.000
6b. Square pattern. Possible techniques	30.8	1.60	14.14	.000
6c. Square pattern. School solution	31.9	1.58	11.63	.000
7a. Food cost. Resolution	32.7	1.84	10.86	.000
7b. Food cost. Arithmetic solution	12.1	1.30	7.63	.000
7c. Food cost. Algebraic solution	10.0	1.17	7.24	.000
8a. Interpreting expressions	22.6	1.27	11.27	.000
8b. Problem statement	20.9	1.30	10.16	.000
9a. Graphic functions. Explanation	75.0	1.54	8.29	.000
9b. Graphic functions. Object recognition	30.5	1.41	10.79	.000
9c. Graphic functions. Curriculum	43.0	1.52	6.49	.000
10a. Linear functions. Statements	16.2	1.35	8.46	.000
10b. Linear functions. Algebra recognition	10.6	1.10	7.61	.000
AVERAGE DIFFICULTY LEVEL	32.5	1.43	35.85	.000

Source: Authors

In the second column, the discrimination index of the items is included, namely the difference in means between the low performance group (percentile 33) and the high performance group (percentile 66) resulting in the proper discrimination of all items. ($p < .001$).

Reliability and validity

The reliability coefficient obtained for the scale (Cronbach's alpha), applied to the sample of 597 students was .783 (> 0.7), considered *acceptable* (Hair, Anderson, Tatham & Black, 2010; George & Mallery, 2003).

In order to obtain proof of the construct validity of the instrument, a factor analysis of main components and *varimax rotation* (Kaiser, 1958) were performed, after verification of the adequacy of the data to this type of analysis. Eight factors were obtained in result. Table 4 shows the rotated component matrix, in which all factor loadings below.33 were eliminated to enable a better analysis of the results.

Each factor is linked to the items included in each task, thus reinforcing the idea that there is a close relationship between mathematical knowledge and specific didactical knowledge involved in the corresponding mathematical activity. There is a high correlation between factor and task in the 8 factors, except 4 and 5.

Factor 4 basically matches task 4 (hexagonal pattern) except for item 2b, which is also included in this factor; item 2b requests identification of properties involved in the equality used as equivalence. Component 5 assembled items from three tasks: Task 8 (interpretation of algebraic expressions and problem statement), Task 3 (sum of three consecutive numbers, types of explanation which may be given by students), item 2a (explanation of the use of equality as equivalence). A possible explanation for this grouping is that in the three items (2a, 3 and 8) the student is expected to analyse a task, support their answer or interpret its meaning, i.e. meta-mathematical tasks involving not only a high degree of abstraction but also a wide range of didactical resources for the analysis of the corresponding mathematical activity.

The remaining factors can be described as follows:

- Factor 1: Didactic–mathematical knowledge related to the mathematical activity of recognising geometric patterns.
- Factor 2: Didactic–mathematical knowledge related to the modelling of problems with arithmetic – algebraic structure.
- Factor 3: Didactic–mathematical knowledge related to the interpretation of linear functions.
- Factor 6: Didactic–mathematical knowledge related to linear functional modelling.
- Factor 7: Didactic–mathematical knowledge related to the use of equality as equivalence.
- Factor 8: Didactic–mathematical knowledge related to the knowledge of arithmetic – algebraic properties of the decimal system.

These results support the idea that learning in didactics of mathematics should be separated from other mathematical contents through a *teaching in action* (Wilhelmi, 2005) that is based on the needs arising from prospective teachers' mathematical and didactical progress.

TABLEA IV. Rotated component matrix

ITEM. Descriptor	Component							
	1	2	3	4	5	6	7	8
1a. Equality arithmetic result. Explanation							.842	
1b. Equality arithmetic result. Interpretation							.719	
2a. Equality equivalence. Explanation					.364		.474	
2b. Equality equivalence. Properties				.463				
3a. Sum three numbers. Generalisation					.505			
3b. Sum three numbers. Type of reasoning					.700			
4a. Partial sum. Solve and explain								.678
4b. Partial sum. Algebraic solution								.647
4c. Partial sum. School solution								.708
5a. Hexagonal pattern. Two terms				.514				
5b. Hexagonal pattern. Algebraic generalisation				.802				
5c. Hexagonal pattern. Type of algebraic objects				.750				
6a. Square pattern. General Solution	.770							
6b. Square pattern. Possible techniques	.806							
6c. Square pattern. School solution	.791							
7a. Food cost. Solution		.824						
7b. Food cost. Arithmetic solution		.771						
7c. Food cost. Algebraic solution		.687						
8a. Interpreting expressions					.521			
8b. Problem Statement					.541			
9a. Graphic functions. Explanation			.712					
9b. Graphic functions. Object recognition			.700					
9c. Graphic functions. Curriculum			.809					
10a. Linear functions. Statements						.844		
10b. Linear functions. Algebra recognition						.844		

Method: Analysis of main components. Rotation method: Varimax with Kaiser normalisation. Source: Authors

Comparative analysis of students' knowledge

Table V presents the means and standard deviations of the variable «Total Score» and the sample size in the different universities. To facilitate comparison and interpretation of the total and partial scores, the scale was changed to the interval [0, 10]. In figure I there is a comparison of the frequency distributions of the total score obtained in the five universities using box plots.

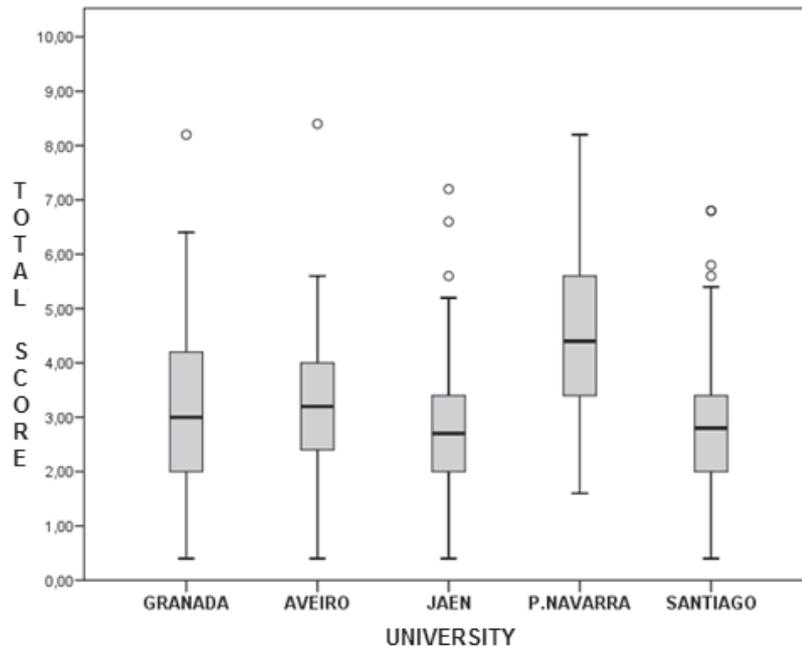
TABLE V. Average scores and standard deviations of the total score per university

University	n	M	DT
Aveiro	66	3.17	1.44
Granada	91	3.25	1.34
Jaén	230	2.77	1.08
Pública de Navarra	129	4.43	1.46
Santiago	81	2.85	1.32
TOTAL	597	3.25	1.43

Source: Authors

In all cases, the average score is below 5. However, there are significant differences in the mean scores obtained in the different universities ($F = 37.05$, $p = .000$) namely the Public University of Navarra (Figure I). Moreover, unlike other universities, in Public of Navarra there is no register of outliers, therefore, if they were suppressed in the other universities differences in means would enhance.

FIGURE 1. Comparison of the total score using box plots



Source: Authors

The correlation coefficient between the average scores of the universities and the cut-off grade to enter the corresponding Faculty is .27, whereas correlation with the admission rate is -.07. In both cases, the relationship is not statistically significant at a 95% confidence level. Additionally, there were no statistically significant differences in average scores between the group of subjects who answered the questionnaire in their last year of training ($M = 3.20$; $SD = 1.45$; Aveiro and Granada) and those who answered it in the beginning ($M = 3.26$, $SD = 1.39$; Jaén, Public University of Navarra and Santiago). The result of the t test comparing independent samples was not significant ($t = 0.46$; $p = .644$). A plausible explanation for this result is that the algebraic contents are not explicitly studied in any of the universities.

Analysis of partial variables: algebraic and didactical knowledge

The analysis of variance used to study the differences in means in the total score variables and the partial variables between different universities (all transformed to the interval $[0, 10]$), suggests that there are significant differences in all variables, as the p -value is $<.001$ in all cases (Table VI).

TABLE VI. Descriptive statistics and comparison of means between universities

Variable	M	SD	F	p
A1_ALG: Common and advanced knowledge in algebra	4.14	1.91	18.54	<.001
A2_EST: Knowledge of equations and relations	3.32	1.60	33.78	<.001
A3_FUN: Knowledge of functions	2.98	1.73	26.77	<.001
A4_MOD: Knowledge of modelling	3.71	2.24	8.54	<.001
CD1_EPI: Knowledge of epistemic aspects	1.49	1.42	29.23	<.001
CD2_COG: Knowledge of cognitive aspects	4.63	2.05	26.56	<.001
CD3_INS: Knowledge of instructional aspects	2.38	2.00	21.84	<.001
TOTAL: Total score	3.25	1.44	37.05	<.001

Source: Authors

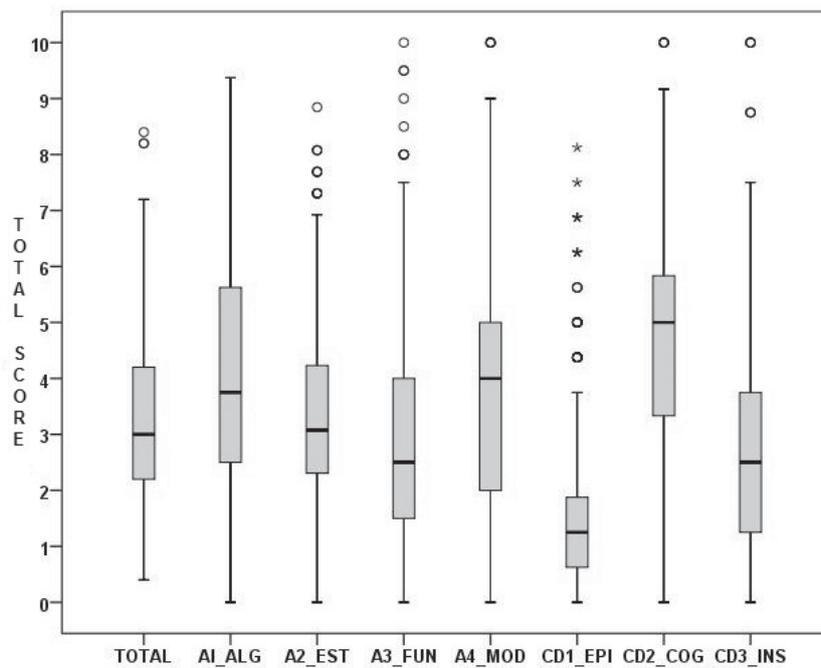
Scores below 5 highlight the difficulties faced by students answering questions of epistemic nature (CD1_EPI), i.e. those that require specialised knowledge of algebraic content.

Figure II shows the frequency distribution of the total score and the 7 partial variables using box plots. Figure II shows there are outliers for all variables, except for A1_ALG. If they were suppressed from the calculations the mean values would be even lower than those presented in Table IV. This finding supports the thesis indicating deficient didactic-mathematical knowledge and the need to take concrete measures to address the existing weaknesses.

Figure III enables the analysis of the differences in student knowledge, according to constructed partial variables. The same graph presents the means of scores obtained for the 7 variables according to the five

universities. There is an existing trend in each university, i.e. if a university's score increases when moving from one variable to another the remaining universities' scores also increase. Similarly, if a university's score decreases when moving from one variable to another the others also decrease, given that the polygonal line of the line graph for each university rarely intersects, a fact that confers coherence and rigour to the study.

FIGURE II. Comparison of the distribution of the different variables using box plots



Source: Authors

When analysed, these variables show a differentiated behaviour at the Public University of Navarra, which presents higher score in all variables, in addition to no intersection being registered with the line graph of the remaining universities. This fact is consistent with the comparison of the

total score displayed in the box plots, contributing to the internal validity of the results.

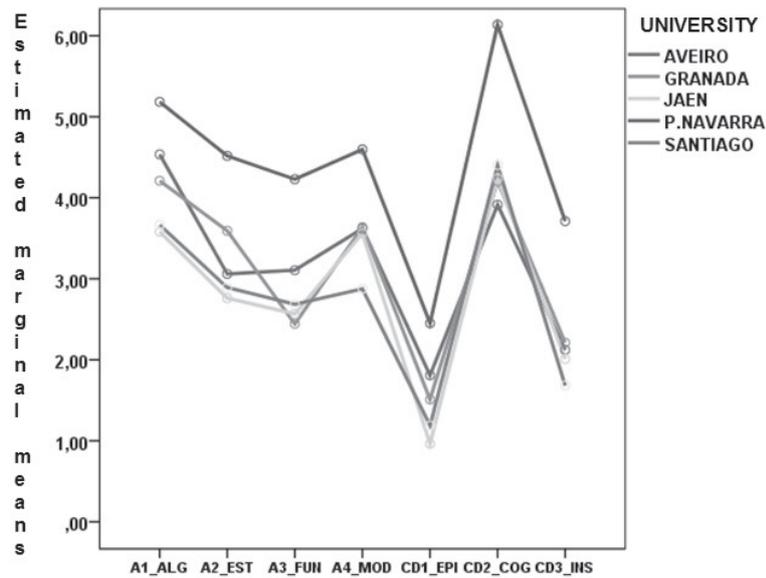
Analysis of algebraic knowledge

The analysis of the level of algebraic knowledge was done according to the following variables:

- A1_ALG: Common and advanced knowledge in algebra.
- A2_EST: Knowledge of equations and relations.
- A3_FUN: Knowledge of functions
- A4_MOD: Knowledge of modelling.

The mean score obtained in the mentioned variables was insufficient. The best result was obtained in A1_ALG (4.14 points) probably because its items are of a procedural nature, i.e. the student must carry out a series of calculations and come up with a result. Following is A4_MOD (3.71 points), which shows the highest degree of dispersion (highest standard deviation), although it is an even dispersion (few outliers). The items in this variable refer to simple models studied in secondary school. Then follows the score in A2_EST (3.32 points). This variable includes the structures, relationships and properties of mathematical framework, which are not mastered by all students. Finally, the lowest score was obtained in A3_FUN (2.18 points), which has many high outliers, justifying the conclusion that some students master the subject whereas the majority has clear difficulties.

FIGURE III. Means of scores obtained in the partial variables



Source: Authors

Analysis of didactical knowledge

The analysis of the level of didactical knowledge was carried out according to the following variables:

- CD1_EPI: Knowledge of epistemic aspects
- CD2_COG: Knowledge of cognitive aspects.
- CD3_INS: Knowledge of instructional aspects.

The mean scores obtained in these variables are also insufficient. The best score was obtained in CD2_COG (4.63 points), with the second best score if we consider all 7 variables. It is evenly distributed, showing few outliers: 50% of students are within a score (4-6) and the remaining 50% are evenly distributed in the two whiskers (box plot).

The other two variables obtained the lowest score, even considering the seven variables. The lowest score was obtained in CD1_EPI (1.49 points). Keeping in mind the box plot, 75% of students have scored less than 2 and 25% have scored between 2 and 8, which results in many outliers. One possible explanation is that the questions of the items included are not common in pre-university education, so students are not used to answering such questions.

Lastly, CD3_INS (2.38 points) is second in the lower scores, indicating our students' lack of instructional practice. 75% of students have scored less than 4, the remaining 25% were rated between 6 and 10, with few outliers. There are no differences at the level of the period in which the questionnaire was administered in the universities, suggesting that the existing difficulties are not resolved throughout their training path.

Summary and implications for teacher training

This paper provides meaningful information regarding the level of knowledge of a significant sample of prospective primary education teachers on elementary algebraic reasoning (EAR). Therefore, it supports previous studies on the nature of EAR (Godino et al., 2014) and the development of measurement instruments (Godino et al., 2015).

Overall, we unveiled major weaknesses in the knowledge of prospective teachers who are currently studying in Spain under the Curriculum established in 2010 (MEC, 2007), in accordance with the guidelines of the European Higher Education Area (Bologna plan). Although the Teacher's Degree in Primary Education has included a fourth year (which meant going from 180 to 240 ECTS), it has not substantially altered the general and psycho-educational profile of teacher training. There is no intention to question the current orientations for the teacher-tutor, who is mainly responsible for teaching Language, Mathematics and Natural and Social sciences, but to highlight the urgent need to ensure training in the various subjects thus enabling the teacher to successfully face the existing epistemological challenges.

The low scores obtained by the Portuguese students studying under a plan of studies which is quite different from the Spanish one, supports the belief that the development of didactic-mathematical knowledge on EAR requires special attention.

Rico et al. (2014, p 58) state that the results of the TEDS-M study point out the need to place greater emphasis on the topics taught in didactics of mathematics and broaden the topics taught in school mathematics: this should be taken into account in the future at both institutional and training levels. In this paper there is experimental data supporting these studies. As a result, the present situation is clearly unsatisfactory, at least regarding algebraic content, despite the implementation of new programmes based on the guidelines of the European Higher Education Area.

The subsequent analysis of the TED-M results by Gutierrez-Gutierrez Gomez & Rico (2014), regarding didactical knowledge on numbers of prospective Spanish primary education teachers trained according to the 1991 plan, highlights weaknesses at the level of such knowledge, which «should be corrected in forthcoming programmes» (p. 296). The results of our study, although it is centred on didactic-mathematical knowledge regarding algebra using our own instrument, prevent us from being optimistic about the improvement of teacher training with the new programmes.

In this paper we have tried to explain the differences in mean scores obtained in the five participating universities by means of its potential relationship with the cut-off required for admission and the admission rate. Although the correlations differ from zero (positive with the cut-off point and negative with the admission rate), they were not significant. In future developments of this study, information should be gathered regarding other potential explanatory variables, namely the students' admission grade, or the measurement of academic achievement in mathematics in previous courses.

On the other hand, from a methodological point of view, it is worth mentioning that the samples were chosen purposefully. This does not decrease the representativeness of the groups in the participating universities, as students are distributed randomly, so it is possible to assume their representativeness with respect to the universities attended. However, a future development of this study, concerning the evaluation of prospective primary teachers' level of knowledge on EAR, should include a wider range of universities explicitly controlling the random and representative distribution of selected samples.

Lastly, the variable CD3_INS, based on prospective teachers' knowledge of instructional aspects, shows very low scores. This hints

towards the need to include didactical and pedagogical strategies in mathematics to enable TDPE students to predict their pupils' behaviour and come up with interventions for the management and control of the study process. In many cases, these activities depend on the number of students per group, which prevents the use of more participatory and collaborative methods, consistent with the basic assumptions of the new European Higher Education Area.

As future lines of work in the evaluation problem of didactic – mathematical knowledge on algebra and its didactics, and under the DMK-EAR model, we can mention the design of items relating to the facets:

- *Affective*, that is, knowledge of future teachers on attitudes, motivation, emotions, beliefs and their influence on the learning of elementary algebra.
- *Ecological*, curricular aspects and links to other contents.
- *Mediacional*, using technological resources in teaching algebra.

This last aspect is closely related to the TPACK⁴ models that identify the types of knowledge that teachers need to master to integrate technology in an effective way in their teaching.

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⁴ Technological Pedagogical Content Knowledge. The TPACK approach extends the theoretical notion introduced by Shulman (1986) of «pedagogical content knowledge» (PCK). A brief description of TPACK is available on the website (<http://www.tpack.org/>), where there is access to specialized literature.

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Appendix: DMK-EAR questionnaire

1. Consider the following question posed to a pupil of the first cycle of primary education:

What number should be placed in the box so that equality is true?

$$8 + 4 = \underline{\quad} + 5$$

A pupil answers that the number is 12,

- Explain the possible reasoning that led the pupil to give that answer.
- Which interpretation of the sign = is being done by the pupil?

2. A pupil was asked to indicate whether the expression « $13 + 11 = 12 + 12$ » is true or false.

The pupil answers the following:

It is true because we subtract one to twelve and add it to the other twelve, the result is what is there (on the left).

- a) Explain the reasoning that could lead the pupil to come up with this response.
- b) Which properties of addition led the pupil to justify their response?

3. A pupil made the following hypothesis: *«I add three consecutive natural numbers. If I divide the result by three I always get the second number»*

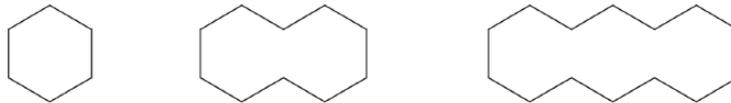
- a) Is the statement valid for all natural numbers? Why?
- b) In your opinion, what kind of justification could a primary school pupil give to this hypothesis?

4. Carefully analyse the following sum, and determine the number representing each letter. Consider that each letter has a different value.

$$\begin{array}{r}
 A \ B \ C \\
 A \ B \ C \\
 + A \ B \ C \\
 \hline
 2 \ A \ C \ C
 \end{array}$$

- a) What are the numerical values of A, B and C? How do you know they are correct? Explain your reasoning.
- b) Can you solve the task using an algebraic procedure? What would that resolution be and which algebraic concepts would be used?
- c) What kind of response and justification do you think an elementary school student could give to this problem?

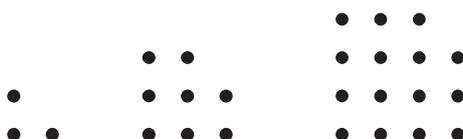
5. Consider the following sequence.



- a) Represent the next two terms of the sequence and indicate the number of segments needed to build each one of them. Explain how you did it.

- b) How would you change the statement of the task to hint a resolution procedure which involved algebraic knowledge?
- c) What would be the algebraic knowledge involved?

6. Consider the following sequence of three shapes defined by dots:



- a) Determine the number of dots that the shape placed on the twenty-fifth (25th) position of this sequence will have, assuming it continues with the same rule of formation. Support your answer.
- b) Indicate techniques or different ways to solve the problem.
- c) Do you consider that this task could be proposed to students in the 3rd cycle of primary school? How could they reach a solution?

7. A pupil received a certain amount of money to eat for 40 days from his parents. However, he found places where he could save 4 euro a day on food. Thus, the initial budget lasted 60 days.

- a) How much Money did he receive?
- b) Can you solve the problem using only arithmetic procedures? How?
- c) Can you solve the problem using algebraic knowledge? How?

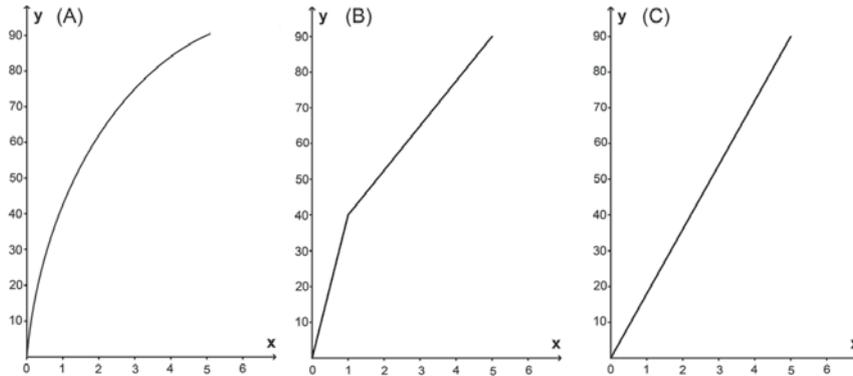
8. Analyse the following expressions and answers:

1. $4x + 5 = 25$
2. $y = 2x + 1$
3. $P = 2c + 2l$

- a) Describe your interpretation of each of the above expressions.
- b) Come up with three problems that may be proposed to elementary students and whose solution leads to these expressions.

9. To fill a container with a maximum capacity of 90 litres with water a faucet whose flow is constant and equal to 18 litres per minute is used.

- a) Indicate which of the three graphical representations corresponds to the situation described above, being that the X axis represents time in minutes and the Y axis the volume of water in litres.



- b) What mathematical knowledge or other type of knowledge is used to solve this task?
- c) Do you consider that this task could be given to primary school children? If so, what cycle? Support your answers.

10. A teacher presents the following problem to his pupils:

At a store they sell each kilo of pears for 2€ and charge 10 cents per bag.

How much would a bag with 4 kg of pears cost?

- a) Come up with a variant of the problem which could be used to introduce linear functions. Suppose each bag carries 4 kg.
- b) Solve the problem you stated and name the algebraic skills used.

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NIPO línea: 030-15-016-X

NIPO lbd: 030-15-017-5

ISSN línea: 1988-592X 0034-8082

ISSN papel: 0034-8082

www.mecd.gob.es/revista-de-educacion